THE ROLE OF REPRESENTATION IN TEACHER UNDERSTANDING OF FUNCTION

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It is becoming widely recognised that teachers' content knowledge has an important influence on their pedagogical content knowledge, and hence on the learning of students. In secondary schools function is one of the fundamental concepts of mathematics. This paper considers the understanding of function exhibited by a group of teacher trainees in response to various representational presentations. The results show that there is a wide range of differing perspectives on what constitutes a function, and that these perspectives are often representation dependant, with a strong emphasis on graphs.

INTRODUCTION

The idea that a teacher's content knowledge base will influence the quality of the understanding that students develop in an area of mathematics has received support from research findings (Ball & McDiarmid, 1990) and curriculum reform documents (National Council of Teachers of Mathematics, 2000). This not particularly surprising since one might expect both lesson goals and structures to be contingent on teacher understanding of the subject matter. After introducing the concept of pedagogical content knowledge for these types of activities by teachers, Shulman (1986) and, later Leinhardt (1989), went on to link the development of mathematics teachers' content and pedagogical content knowledge, suggesting the existence of links between content knowledge and explanations and representations generated during teaching. Confirmation of this was found in a study of the effects of content knowledge in algebra by Menzel and Clarke (1999, p. 371), which noted that teachers with a weak content knowledge "lacked the detailed knowledge needed to both identify specific student difficulties and construct situations that might assist students to overcome their difficulties." The extent of the difference which teachers can make is shown in the large-scale study of Sullivan and McDonough (2002). Their conclusion about the influence of teachers on learning was that "The data presented here suggested that the differences [in improvement of student learning] between the most effective and least effective teachers are substantial." (p. 255).

Function is a fundamental concept of school mathematics and hence, a teacher's content knowledge of function is likely to be crucial to providing a positive learning environment for much of secondary school mathematics. Even's (1998) research with college students emphasised the importance of representations in understanding of function, finding that students had difficulties in flexibly linking different representations and finding a link with pointwise and global approaches to function problems. Indeed in a research project where the relationship of an experienced teacher's conceptions of function to his practice was examined, Lloyd and Wilson (1998) found that he valued explorations of multiple representations of problem situations and that these offered students increased opportunities to understand. Furthermore, his content knowledge structures did influence his teaching, so that the "teacher's comprehensive and well-organised conceptions contribute to instruction characterised by emphases on conceptual connections, powerful representations, and meaningful discussion." (p. 270).

The contrast between the simplicity of the function concept and the complexity of its manifestations, and the concept images it may evoke (Vinner, 1983) has been well described by Akkoc and Tall (2002). They found that some students were unable to see and apply the fundamental (simple) definition of function, instead relying on almost arbitrary aspects of examples they focussed on. Comparing the function concept maps of eight professors having PhDs in mathematics with those of twenty-eight university mathematics students, Williams (1998) found that the student maps portrayed an emphasis on minor detail, such as the variable used, algorithms, and the idea that functions are equations. In contrast she found that "none of the experts demonstrated the students' propensity to think of a function as an equation. Instead, they defined it as a correspondence, a mapping, a pairing, or a rule." (*ibid,* p. 420). Chinnappan and Thomas (2001) found that their trainee teachers had a strong tendency to think of functions graphically and procedurally, separating algebra from functions (which they saw as graphical) in their thinking, and displaying gaps in their knowledge of function.

Hence this research sought to understand further prospective teachers' thinking about functions and its relationship to function representations and the formal concept.

METHOD

This research comprised a case study analysis using a group of thirty-four pre-service secondary mathematics teacher trainees at The University of Auckland. The teacher training at this institution is a graduate programme and so all had a degree with a substantial component of mathematics and had done some teaching. Each of the teachers was given a questionnaire comprising 13 questions. In each case they were presented with either an algebraic, graphical, ordered pair, or tabular representation and were asked to say whether or not it could be seen as a way of representing a function, giving a reason for their answer (See Figure 1 for a summary of the representations). Some 'grey' areas and other key properties of functions were deliberately targeted in the questions. Hence the choice of representations presented to the teachers included the issues of values where the function is not defined (e.g., Q1), the lack of an explicit statement on domain (e.g., Q6, 8), absence of information on whether y or z is a function (e.g., Q4, 8), and acceptance of a function of two variables (Q9), etc.

Another specific purpose behind the choice of these representations, apart from a consideration of the role of the representation itself, was to examine the level of the perceived need for an equation or formula with two explicit variables in order to have a function. During informal discussions with teachers, many were not happy to describe the algebraic form x^2 as a function, stating that it had to be written as $y=x^2$ before it could be a function. However, it emerged that they were comfortable with seeing forms such as e^x, $\sin x$ and $\cos x$, etc as functions. The idea of requiring two variables appears to arise from the requirement in the formal definition of function of specifying the domain and co-domain before the relation or 'rule' defining the function is presented. The motivation in the research was to try and understand and document aspects of the subjects' concept image of function, and how it is influencing their mathematical thinking, and hence their teaching.

1. $f(x)=\dfrac{e^{-x}\ln(1+x)}{\sin 2x}$

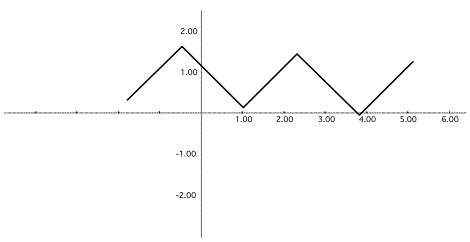
2. $(x, 2x)$, where x is a real number

3. $\begin{bmatrix} p & 0 \\ 0 & 0.5 \end{bmatrix}, \quad p \in Z$

4. $\dfrac{d^2y}{dx^2}$

6. {(1,3), (2,5), (3, 4)}

5.



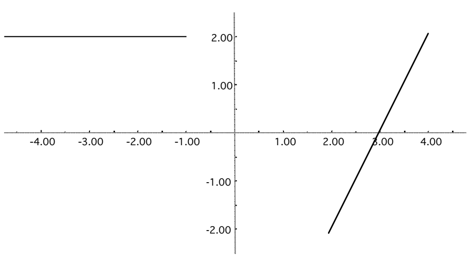
12.

x	y
1	9
2	16
3	25
4	36
5	47

7. $g(y)=\begin{cases} 1 & y \text{ rational} \\ y & y \text{ irrational} \end{cases}$

8. $\lim\limits_{h \to 0} \dfrac{f(z+h)-f(z)}{h}$

10.



9. (a, b) $\to$ a + b

11. $\int_1^x \frac{1}{t}\,dt \quad x \in R$

13. $\{(x, y) \mid x \in \mathrm{N}, y = 1\}$

Figure 1. The representational formats used in the questionnaire.

RESULTS

As may be seen from Table 1 there was no question on which the teachers were unanimous about whether the given format represented a function or not. In every case except question 3 (which was split 41% Yes, 47% No) there was a majority considering that it was a function, although in many questions there was a significant minority, between 23.5% and 29%, disagreeing. If the teachers' decisions were based solely on the application of a basic function definition then one would expect a more tightly positioned distribution, with values something like those in Q5, Q10, and Q12.

However, on closer inspection it became clear that as well as the lack of unanimity, there were also some interesting reasons given for the answers, as noted in the responses below. As expected, based on the example x^2 mentioned above, some of the replies

indicated that the representation could not be a function unless there was an equation, or

Table 1: *Number of Teachers Describing Each Representation as a Function*

Question Number	Response to Whether a Function or Not (N=34)		
	Yes	No	No Answer
1	23	9	2
2	26	8	0
3	14	16	4
4	21	12	1
5	30	4	0
6	24	10	0
7	21	9	4
8	21	13	0
9	16	11	7
10	29	3	2
11	21	8	5
12	30	4	0
13	19	10	5

two variables explicitly given. For example, responding to Qs 2 and 6, which had ordered pairs, T26 (teacher 26) wrote "no variable" as the reason for it not being a function. However, for Q13 which also has ordered pairs but in a set format, he was happy to accept it as a function "mapping y=1 for all x □ N."

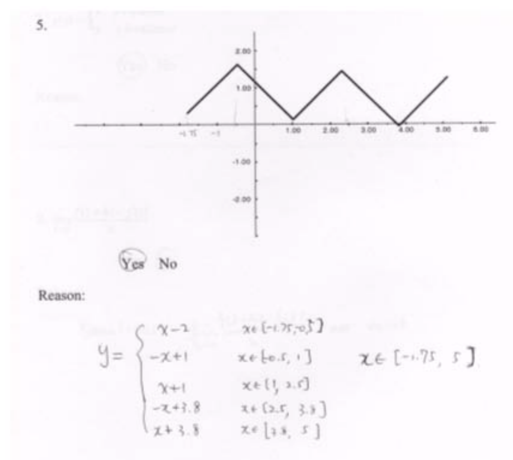


Figure 2. Need for formulas to define a function.

Similarly, T8 wrote that Q2 was not a function, because it "only contains x values", yet was happy to describe Q11 as a function, since "each x value has a corresponding y", even though there was no y given in the representation. For Q7 T1, explaining why it is not a function wrote "It is not a formula or relation of 2 or more variables but it gives only one variable and the value of the variable is determined by a number and itself." Thus she did not see the notation $g(y)$ as sufficient for specifying the range, but wanted another explicit variable in the equation. T3 took the idea of wanting an equation for the

function to extremes in Q5 providing a formula for each of 5 sections of the split domain function (see Figure 2). In Q13, where two variables were given but in the context of an ordered pair representation, T7 said that it was not a function because "It is merely a set of 2 values, and they do not affect the value of others." However, for Q6 she said that it was a function because in the given domain "Each x has a corresponding y value." This may indicate that it was not the ordered pair representation that was the problem for her in Q13 but the format or the context of the notation.

The tabular representation in Q12 proved to provide a context in which a number of the teachers were looking for a formula that would fit the table. However, it had been arranged that the value of y at x=5 was 47 not 49 in order to ascertain how they would cope with this if they wanted a formula. While some teachers, such as T1, T3, and T26 either 'corrected' it to 49, assuming it was an error, or missed or ignored it, so that it fitted the formula y=(x + 2)2 that they had modelled to the data, others such as T28 said it was not a function because "What's the eqn [equation] used to get the value of y?" T19 also thought that it was not a function unless the 47 was made into 49 so that a "formula for it" could be modelled (see her reasoning in Figure 3). Hence the table clearly evoked the concept of a formula or equation for function in a number of the teachers, supporting the observation of Williams (1998).

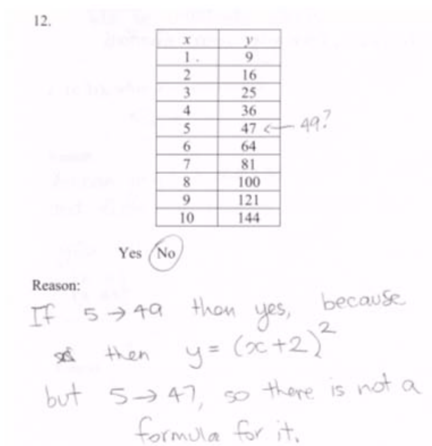


Figure 3. Algebraic formula/equation needed for a tabular representation of function.

In contrast, eight of the teachers used an argument for Q12 based on some form of reasoning to do with mapping of variables, more akin to the experts in Williams' (1998) study, to establish it as a function. These were generally more successful in arguing for a function, making comments such as: "Every element *x* in the domain is linked to only one element in the range." (T6); "each *x* has a corresponding *y*" (T8); "Each *x* value has only one *y* value." (T14); "each *x* value has only 1 corresponding *y* value." (T20); "One *x* only one *y*." (T24); "There is a relation that maps each value of *x* to *y*" (T30); and "Each *y* is different. So for each *x* there is at most one *y* i.e. it's a function." (T16). While these are not all complete arguments (e.g., saying both that every *x* has a value and each *x* has only one *y* value), or have extraneous detail (e.g., 'each *y* is different') they *are* using aspects of an informal definition. However, they did not always correctly apply this line of reasoning based on the fundamental definition of function, with more emphasis being placed on the mapping nature than on the actual formal definition. For example, T9 wrote

"For every $f(x)$ there is an x value", which makes it an onto relation, but not necessarily a function.

Not surprisingly the table representation in this question was linked to a graph by several teachers. T2 and T29 stated that it was a function because it was a "list of points in an x,y plane" and "Because this table of values can be plotted on a set of axes". They seem to have constructed the idea that any planar graph drawn from a table represents a function. The graphical link also evoked the erroneous consideration that the function must be continuous, with T23 saying that it was a function because "there exists such function such that connects all of these points" and he drew a continuous graph. This is no doubt linked to the idea of getting a formula or equation to model the data, since in their experience most of the graphs of algebraic functions (often polynomials) would have been continuous. Nine of the teachers gave no reason for their answer on Q12.

The concept that we have a function if a graph can be drawn arose in other questions too. T8, for example, wrote that Q4 was a function because "you can draw a graph" and gave similar reasons for Q10 and Q12 (where the graphs had been drawn). It may be that a lack of experience of graphs of non-functions has caused them to forget that not all 2-dimensional graphs represent functions.

Some students were very strongly constrained to a graphical perspective on function, reducing most examples, where they could to a graph. T33, for example, attempted to use the 'vertical line test' wherever she could, since this was clearly the dominant idea of function for her. So for Q2, Q5, Q9, Q10, and Q13 she employed this method, stating in Q2, "increasing straight line cuts vertical line test." In Q13 (see Figure 4) she managed to draw the graph with discrete points (a 'point graph' as she calls it) and see that any vertical line would cut it at most once. However, there were some problems applying this test, and she struggled to use it for Q3, stating "I'm not sure what this vector looks like on 2 dimensional plane □ therefore not a function."

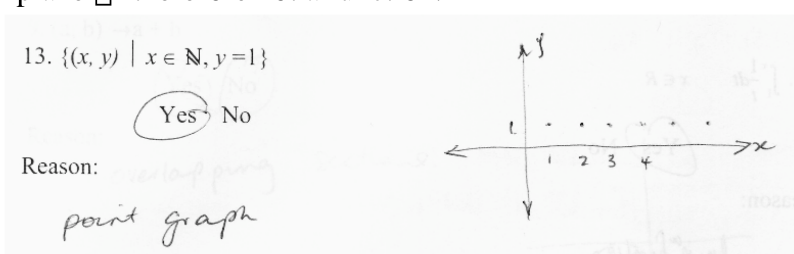


Figure 4. Relating an ordered pair and a graphical representation.

A reliance on graphs was also seen in the answers of T20, who drew graphs for Q2 and Q6 (see Figure 5) but wrestled with questions such as Q3 (not answered, and a '?' placed by it), Q7 and Q9 (both not answered) where the graphs proved too difficult. It is interesting that in Q2 he linked the ordered pairs given (in terms of x) to a graph and then to the equation $y=2x$ in order to make a decision. However, in Q6 he linked the ordered pairs (without the x present) to a graph and then used reasoning based on a correspondence between variable values. In each case he accepted it as a function. Clearly the portion of the function concept image evoked varies subtly depending on slight variations in the representational content.

T20 was among 4 of the teachers (T1, T16, T20 and T22) who thought that the graph in Q10 was not a function since it was discontinuous.

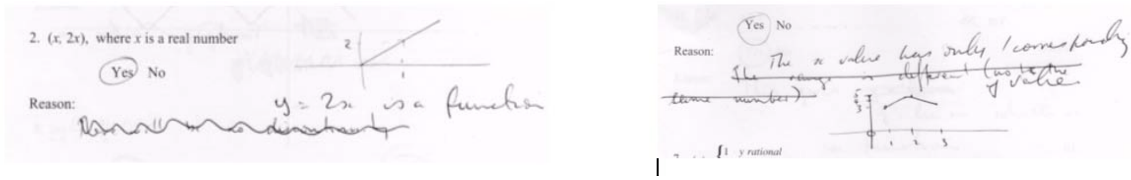


Figure 5. Use of graphs leading to differing reasoning to decide on function status.

No teacher mentioned the fact that the function in Q5 was not differentiable. This may be because they thought it irrelevant or that they did not consider it at all.

There was some evidence of a procedural approach to the learning of function among these teachers. Attempting to learn in an instrumental manner can easily lead to errors. In one case T29 had tried to learn the vertical line test for a graphical representation of a function, but had mis-remembered it as a horizontal line test. As Figure 6 shows she used it in Q10 to say that it would not represent a function "if a horizontal line crosses both lines", and so was not a function.

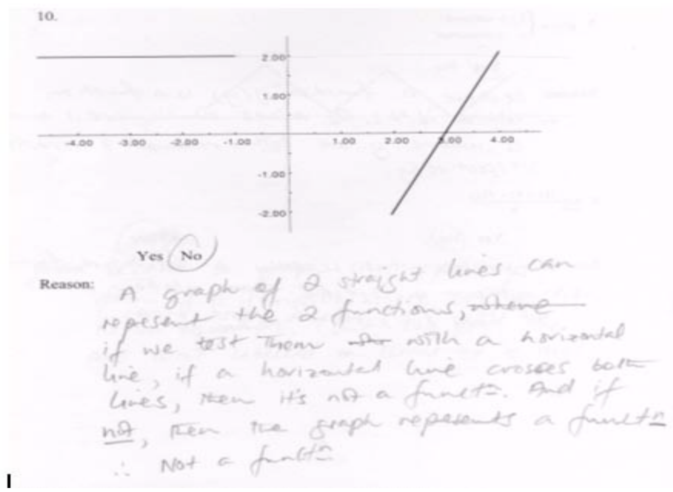


Figure 6. Instrumental learning of a rule goes wrong.

CONCLUSION

There is no doubt that for some of the teachers the questionnaire highlighted the shaky nature of their hold on an understanding of function, including some principal elements of the function concept missing from their concept images, and this probably led to the lack of unanimity. Supporting this, when the teachers were invited to make a general comment at the end of the questionnaire, T9 wrote 'I suppose I realised how unsure I am about what makes a function a function." and T29 added "Thanks for reminding me what a function is. But I still couldn't remember what it is exactly."

The most success in terms of giving supportable reasons for whether the representations were of functions came from those who, like the experts in Williams' (1998) study based their thinking on relationships or mappings between values of variables, even though they did not directly refer to a formal definition of function. There is also some evidence here that rather than seeing function as a concept that crosses representational boundaries some of the trainee teachers are engaging in shifting their concept image focus depending on the representation they are engaged with, as reported by Lauten, Graham and Ferrini-

Mundy (1994). Throughout the questions the graphical perspective had a strong dominance for a number of the teachers. While visual imagery is often a useful asset for assisting mathematics learning, Aspinall, Shaw and Presmeg (1997) have described how an uncontrolled use of it can have negative implications for application of functions in calculus, and Chinnappan and Thomas (2001) present examples of this too. It would seem that the data presented here support the view that for many teachers the graphical representation of function is becoming dominant to such an extent that it could hinder a growth in inter-representational understanding. Certainly the teachers, and hence their students, would benefit from development of stronger inter-representational thinking about function.

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